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Lurie Equations and Equivalent Hamiltonian Systems

M. G. Yumagulov[*,a] **and L. S. Ibragimova**[*,b]

[*] *Ufa University of Science and Technology, Ufa, Russia*
e-mail: [a] *yum_mg@mail.ru,* [b] *lilibr@mail.ru*

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Abstract—This paper proposes new approaches to constructing equivalent Hamiltonian systems for linear and nonlinear Lurie equations (differential equations containing the derivatives of even orders only). The approaches are based on the transition from the linear part of the Lurie equation to the normal forms of the corresponding Hamiltonian systems, with a subsequent transformation of the resulting system. This scheme does not require complex and cumbersome transformations of the original equation. The effectiveness of the formulas derived is illustrated by examples.

Keywords: Lurie equation, Hamiltonian system, normal form, equivalence, observability.

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1. INTRODUCTION

Consider the differential equation

$$L\left(\frac{d}{dt}\right)y = M\left(\frac{d}{dt}\right)f(y), \tag{1}$$

where

$$L(p) = p^{2n} + a_1p^{2n-2} + a_2p^{2n-4} + \ldots + a_{n-1}p^2 + a_n,$$
$$M(p) = b_0p^{2m} + b_1p^{2m-2} + \ldots + b_{m-1}p^2 + b_m,$$

are coprime polynomials ($0 \leqslant m < n$) and $f(y)$ is a scalar continuous function. Equation (1) describes the dynamics of a single-loop control system consisting of a linear link with the fractional-rational transfer function $W(p) = M(p)/L(p)$ and a nonlinear feedback with the characteristic $f(y)$; for example, see [1, 2]. Note that equations of the form (1) are often called *Lurie equations*.

The polynomials $L(p)$ and $M(p)$ contain degrees of even orders only. Differential equations of even orders arise in many problems of control theory, the theory of Hamiltonian systems, the theory of integrable equations, spectral theory, etc. In studies of such equations, an important direction is the problem of introducing a Hamiltonian structure to them. The availability of such a structure (as a consequence, the existence of first integrals and various types of symmetries) allows advancing significantly in the analysis of systems dynamics. The issues regarding the existence of a Hamiltonian structure for many types of differential equations and, accordingly, the construction of an equivalent Hamiltonian system for equations (1) in various problem statements were discussed in several research works, e.g., [2–9]. The problem statements below are close to those considered in [10, 11].

In this paper, we present new approaches to studying the above issues. The approaches are based on the transition from the linear part of the Lurie equation to the normal forms of the corresponding Hamiltonian systems, with a subsequent transformation of the linear and nonlinear systems. The results obtained lead to effective algorithms for constructing the Hamiltonian of the system. The results can be applied to analyze the dynamics of systems described by differential equations of even orders as well as the stability and bifurcations of equilibria and periodic solutions of linear and nonlinear Lurie equations.

2. BACKGROUND

We recall some concepts of systems theory, control theory [1, 2, 7, 8], and the theory of Hamiltonian systems [3, 4].

2.1. The Equivalence of Systems

Let $\mathcal{A}$ and $\mathcal{B}$ be two systems described by the input-output-state equations. Assume that these systems have the same space $\mathcal{U}$ of inputs $u(t)$ and the same space $\mathcal{Y}$ of outputs $y(t)$. We denote by $\mathcal{S}$ and $\mathcal{T}$ the state spaces of systems $\mathcal{A}$ and $\mathcal{B}$, respectively.

Systems $\mathcal{A}$ and $\mathcal{B}$ are said to be *equivalent* if, for each state $\alpha \in \mathcal{S}$, there exists a state $\beta \in \mathcal{T}$ such that the outputs of systems $\mathcal{A}$ and $\mathcal{B}$ will coincide for the same inputs $u(t) \in \mathcal{U}$ and vice versa. In this case, we will write $\mathcal{A} \sim \mathcal{B}$.

2.2. On the Observability of Systems

Consider a dynamic system described by the equation

$$x' = Ax + \xi u(t), \quad y = (x(t), c), \tag{2}$$

where A is a square matrix of order n; ξ, $c \in R^n$ are fixed vectors; the symbol (x, c) indicates the inner product of vectors x and c from R^n. In this system, u, y, and x denote the input, output, and state, respectively.

Throughout this paper, vectors will be treated as column vectors unless they are explicitly stated to represent row vectors in a particular formula.

We define a square matrix of order n:

$$D = \begin{bmatrix} c \\ A^*c \\ (A^*)^2c \\ \vdots \\ (A^*)^{n-1}c \end{bmatrix}, \tag{3}$$

where A^* means the transpose of A and the vectors c, A^*c, $(A^*)^2c$, $\ldots$, $(A^*)^{n-1}c$ are row vectors. The matrix D is called the *observability matrix* of system (A.3). System (A.3) is said to be *observable* if $\det D \neq 0$.

2.3. On Hamiltonian Systems

An *autonomous Hamiltonian system* is a dynamic system described by the equation

$$x' = J\nabla H(x), \quad x \in R^{2n}, \tag{4}$$

where

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \qquad \nabla H(x) = \left(\frac{\partial H}{\partial x_1}, \ldots, \frac{\partial H}{\partial x_{2n}}\right)^T, \tag{5}$$

0 and I stand for zero and identity matrices, respectively, of order n, and $H(x)$ is a scalar real smooth called the *Hamiltonian* of system (4).

A *linear autonomous Hamiltonian system* (LAHS) is a system of the form

$$\frac{dx}{dt} = JAx, \quad x \in R^{2n}, \tag{6}$$

where A is a real square symmetric matrix of order $2n$. The Hamiltonian of this system is given by

$$H(x) = \frac{1}{2}(Ax, x). \tag{7}$$

Below, the matrix JA participating in system (6) will be called the *Hamiltonian matrix*. Note its properties as follows:

G1) If the matrix JA has an eigenvalue λ, then the numbers $-\lambda$, $\overline{\lambda}$, and $-\overline{\lambda}$ are also eigenvalues of this matrix, with the same algebraic and geometric multiplicity and the same index.

G2) If the matrix JA has the eigenvalue $\lambda = 0$, then the algebraic multiplicity of this eigenvalue is an even number.

G3) The characteristic polynomial of the matrix JA contains degrees of even orders only.

Each Hamiltonian matrix belongs to one and only one equivalence class of symplectically similar matrices. In each such class, one representative, called the *normal form*, is often distinguished. The kind of the normal form is determined by the properties of the root subspaces of the matrix JA. We refer to [3, 9, 12, 13] for more details on the theory of normal forms and, in particular, the lists of normal forms.

A specific feature of normal forms is that different normal forms may correspond to a given set of eigenvalues with given multiplicities. As an illustration, consider fourth-order Hamiltonian matrices having two pairs of prime pure imaginary eigenvalues $\pm\omega_1 i$ and $\pm\omega_2 i$, where $\omega_1 > 0$ and $\omega_2 > 0$. In this case, there are two kinds of normal forms:

$$JA = \begin{bmatrix} 0 & 0 & \omega_1 & 0 \\ 0 & 0 & 0 & \sigma\omega_2 \\ -\omega_1 & 0 & 0 & 0 \\ 0 & -\sigma\omega_2 & 0 & 0 \end{bmatrix}, \quad \text{where} \quad \sigma = 1 \ \text{or} \ \sigma = -1. \tag{8}$$

In the case $\sigma = 1$, the numbers $\omega_1 i$ and $\omega_2 i$ are called the *eigenvalues of the first kind*; in the case $\sigma = -1$, they are called the *eigenvalues of the first and second kind*, respectively. No symplectic transformations can reduce the normal form with $\sigma = 1$ to the normal form with $\sigma = -1$.

The above properties of Hamiltonian matrices determine many important qualitative characteristics of Hamiltonian systems (linear and nonlinear), such as strong stability properties, stability in the linear and nonlinear formulation, etc.; for example, see [9–15].

As will be shown below, due to this fact, the problem of constructing an equivalent Hamiltonian system for equation (1) may have qualitatively different solutions, namely, the resulting Hamiltonian systems (6) may have different normal forms.

3. THE LINEAR PROBLEM

3.1. The Standard Change of Variables

We discuss the problem of constructing an equivalent Hamiltonian system first for the linear equation

$$L\left(\frac{d}{dt}\right) y = 0. \tag{9}$$

With the standard change of variables

$$z_1 = y,\ z_2 = y',\ \ldots,\ z_{2n} = y^{(2n-1)}, \tag{10}$$

this equation is reduced to an equivalent system in the state space:

$$z' = A_0 z, \quad y = (z, c_0), \tag{11}$$

where $z,\ c_0,\ \gamma \in R^{2n}$, the symbol (z, c_0) indicates the inner product of vectors, and

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 \ldots & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & 0 \ldots & 0 & 1 \\ -a_n & 0 & -a_{n-1} & 0 \ \ldots & -a_1 & 0 \end{bmatrix}, \quad c_0 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}. \tag{12}$$

System (11) is Hamiltonian only for $n = 1$, i.e., when equation (9) takes the simplest form $y'' + a_1 y = 0$. If $n \geqslant 2$, system (11) is no longer Hamiltonian. From this point onwards, we assume that $n \geqslant 2$.

3.2. Constructing the Hamiltonian System

Since the polynomial $L(p)$ contains degrees of even orders only, the roots of the equation $L(p) = 0$ have properties similar to properties G1 and G2 of Hamiltonian matrices. Therefore, the polynomial $L(p)$ with this set of roots can be assigned one or more normal forms with the same set of eigenvalues.

We propose the following construction scheme of an equivalent Hamiltonian system for equation (9).

At the first stage, the roots of the equation $L(p) = 0$ are used to determine possible normal forms of the desired Hamiltonian system. One of the corresponding Hamiltonian matrices JA is chosen.

The second stage is to define a nonzero vector $c \in R^{2n}$ and the Hamiltonian system

$$\frac{dx}{dt} = JAx, \quad y = (x(t), c). \tag{13}$$

Theorem 1. *Equation* (9) *and the Hamiltonian system* (13) *are equivalent iff system* (13) *is observable.*

This theorem can be supplemented by the following result. Let $\tilde{y} = \begin{bmatrix} y \\ y' \\ \vdots \\ y^{(2n-1)} \end{bmatrix}$, where $y^{(k)}$ denotes the kth-order derivative of the scalar function $y = y(t)$.

Theorem 2. *Assume that one of the possible normal forms of JA is chosen according to the properties of the roots of the equation $L(p) = 0$. Assume also that the vector c is appropriately chosen to make system* (13) *observable. Then the change of variables $x = D^{-1}\tilde{y}$, where D is the observability matrix of system* (13), *reduces equation* (9) *to the equivalent Hamiltonian system* (13) *with the Hamiltonian* (7). *In addition, the matrices A_0 and JA are related by the equality $A_0 = D(JA)D^{-1}$.*

The proofs of Theorems 1 and 2 and other main results are postponed to the Appendix.

Remark 1. According to Theorems 1 and 2, for equation (9), the problem of constructing an equivalent Hamiltonian system in normal form may have a nonunique solution. In other words, equation (9) can be reduced, via linear nondegenerate transformations, to qualitatively different Hamiltonian systems of the form (13) in the sense that the corresponding Hamiltonian matrices belong to different equivalence classes of symplectically similar matrices.

Note also that for equation (9), the problem of constructing an equivalent Hamiltonian system with a particular normal form may be unsolvable. This situation arises, e.g., when the equation $L(p) = 0$ has multiple roots. In this case, equation (9) may lead to such normal forms of Hamiltonian matrices for which the corresponding system is unobservable for any vector c.

3.3. A Linear Link with Two Degrees of Freedom

As an illustration, consider the fourth-order Lurie equation

$$y'''' + ay'' + by = 0, \tag{14}$$

where the real coefficients a and b satisfy the conditions

$$a > 0, \;\; b > 0, \;\; d = a^2 - 4b > 0. \tag{15}$$

In this case, all the four roots of the characteristic equation

$$\lambda^4 + a\lambda^2 + b = 0$$

are different and pure imaginary of the form $\pm i\omega_1$, $\pm i\omega_2$, where the numbers $\omega_1 > 0$ and $\omega_2 > 0$ satisfy the equation $\omega^4 - a\omega^2 + b = 0$, i.e.,

$$\omega_1^2 = \frac{a + \sqrt{d}}{2}, \quad \omega_2^2 = \frac{a - \sqrt{d}}{2}. \tag{16}$$

Now we discuss the construction of an equivalent Hamiltonian system for equation (14).

Let us utilize the above scheme. In the problem under consideration, equation (14) can be reduced to two different normal forms of the desired Hamiltonian system, namely, the matrix (8) for $\sigma = 1$ and $\sigma = -1$. With an appropriate choice of the vector $c \in R^4$, it is possible to obtain two qualitatively different LAHSs (13) with the normal form (8) of the matrix JA that are equivalent to equation (14) both for $\sigma = 1$ and for $\sigma = -1$.

To show this fact, let $c = (c_1, c_2, 0, 0)$ be some vector such that $c_1c_2 \neq 0$. Then equation (14) can be reduced to the Hamiltonian system (13) via a linear nondegenerate transformation.

Indeed, to apply Theorem 1, we should establish the observability of system (13) with the normal form (8) of the matrix JA. We have

$$(JA)^*c = \begin{bmatrix} 0 \\ 0 \\ \omega_1 c_1 \\ \sigma\omega_2 c_2 \end{bmatrix}, \quad (JA^*)^2 c = \begin{bmatrix} -\omega_1^2 c_1 \\ -\sigma\omega_2^2 c_2 \\ 0 \\ 0 \end{bmatrix}, \quad (JA^*)^3 c = \begin{bmatrix} 0 \\ 0 \\ -\omega_1^3 c_1 \\ -\sigma\omega_2^3 c_2 \end{bmatrix}. \tag{17}$$

Hence, the matrix (3) takes the form

$$D(c) = \begin{bmatrix} c_1 & c_2 & 0 & 0 \\ 0 & 0 & \omega_1 c_1 & \sigma\omega_2 c_2 \\ -\omega_1^2 c_1 & -\sigma\omega_2^2 c_2 & 0 & 0 \\ 0 & 0 & -\omega_1^3 c_1 & -\sigma\omega_2^3 c_2 \end{bmatrix}, \tag{18}$$

and

$$\det D(c) = \begin{cases} -c_1^2 c_2^2 \omega_1 \omega_2 (\omega_1^2 - \omega_2^2)^2 & \text{if } \sigma = 1 \\ c_1^2 c_2^2 \omega_1 \omega_2 (\omega_1^4 - \omega_2^4) & \text{if } \sigma = -1. \end{cases}$$

This means that $\det D(c) \neq 0$ for $c_1 c_2 \neq 0$ and $\omega_1 \neq \omega_2$. Thus, the matrix $D(c)$ is nonsingular (reversible), and system (13) is observable accordingly. By Theorem 1, equation (14) and system (13) are equivalent. By Theorem 2, the change of variables $\tilde{y} = D(c)x$, where $\tilde{y} = \begin{bmatrix} y \\ y' \\ y'' \\ y''' \end{bmatrix}$, reduces system (13) to the scalar differential equation (14). The solutions $y(t)$ and $x(t)$ of equation (14) and system (13) are related by the equality $y(t) = c_1 x_1(t) + c_2 x_2(t)$.

Thus, an equivalent Hamiltonian system for the linear equation (14) has been constructed. Once again, we underline that equation (14) can be reduced to two different Hamiltonian representations (13) with the normal forms (8). Additional information about the object under study is required for a particular choice of the normal form.

EXAMPLE 1

The planar bounded circular three-body problem is one of the most interesting problems in celestial mechanics; for example, see [13, 16–18]. In the linear statement, the problem of investigating the motion of a small-mass body in the neighborhood of triangular libration points leads to the differential equation

$$y'''' + y'' + \frac{27}{4}\mu(1-\mu)y = 0. \tag{19}$$

Its characteristic equation has the form

$$\lambda^4 + \lambda^2 + \frac{27}{4}\mu(1-\mu) = 0. \tag{20}$$

Following the above scheme, we pass from equation (19) to an equivalent LAHS of the form (13). Let $\mu \in (0, \mu^*) \bigcup (1-\mu^*, 1)$, where $\mu^* = \frac{1}{2} - \frac{\sqrt{69}}{18} \approx 0.0385$. In this case, all the four roots of equation (20) are pure imaginary: $\lambda_{1,2} = \pm\omega_1(\mu)i$, $\lambda_{3,4} = \pm\omega_2(\mu)i$; here

$$\omega_1(\mu) = \sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{1-27\mu(1-\mu)}}, \quad \omega_2(\mu) = \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1-27\mu(1-\mu)}}.$$

Hence, there are two kinds of the normal forms (8). For a particular kind determined from certain considerations, we choose, e.g., the vector $c = (1, 1, 0, 0, 0)$. Using (17) and (18), we construct the matrix $D = D(c)$, which turns out to be nonsingular. Consequently, with the change of variables $\tilde{y} = D(c)x$, equation (19) can be reduced to an equivalent Hamiltonian system of the form (13), and their solutions $y(t)$ and $x(t)$ are related by the equality $y(t) = x_1(t) + x_2(t)$.

Note that according to the analysis of the original three-body problem statement [13], one should take $\sigma = -1$ in the normal form (8).

4. THE NONLINEAR PROBLEM

4.1. Main Results

Now we discuss the problem of constructing an equivalent Hamiltonian system for the nonlinear Lurie equation (1).

As in the linear problem, the first stage is to determine possible normal forms of the linear part of the desired Hamiltonian system using the roots of the equation $L(p) = 0$. And one of the corresponding Hamiltonian matrices JA is chosen.

At the second stage, it is necessary to define a nonzero vector $c \in R^{2n}$ and the linear Hamiltonian system (13). Assume that this system is observable. Let $D = D(c)$ be the corresponding observability matrix.

We define the vectors

$$\gamma = \begin{bmatrix} 0 \\ \gamma_2 \\ 0 \\ \gamma_4 \\ \vdots \\ 0 \\ \gamma_{2n} \end{bmatrix}, \quad \tilde{y} = \begin{bmatrix} y \\ y' \\ \vdots \\ y^{(2n-1)} \end{bmatrix}, \quad \tilde{f}(y) = \begin{bmatrix} f(y) \\ (f(y))' \\ (f(y))'' \\ \vdots \\ (f(y))^{(2n-3)} \end{bmatrix}, \tag{21}$$

where the t-derivatives $y^{(k)}$ and $(f(y))^{(k)}$ of given functions $y = y(t)$ and $f(y(t))$, respectively; the coordinates of the vector γ are given by

$$\begin{gathered} \gamma_2 = \gamma_4 = \ldots = \gamma_{2n-2m-2} = 0, \quad \gamma_{2n-2m} = b_0, \\ \gamma_{2n-2m+2} + \gamma_{2n-2m}a_1 = b_1, \ \ldots, \ \gamma_{2n} + \gamma_{2n-2}a_1 + \ldots + \gamma_{2n-2m}a_m = b_m. \end{gathered} \tag{22}$$

Also, we define a rectangular matrix of order $2n \times (2n-2)$:

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ \gamma_2 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & \gamma_2 & 0 & 0 & \ldots & 0 & 0 \\ & & & & \ddots & & \\ \gamma_{2n-2} & 0 & \gamma_{2n-4} & 0 & \ldots & 0 & 0 \\ 0 & \gamma_{2n-2} & 0 & \gamma_{2n-4} & \ldots & 0 & \gamma_2 \end{bmatrix}.$$

Lemma 1. *Assume that the linear system* (13) *is observable. Then the change of variables*

$$x = (D(c))^{-1}[\tilde{y} - T\tilde{f}(y)] \tag{23}$$

reduces equation (1) *to the system*

$$x' = JAx + \xi f(y), \quad y = (x(t), c), \tag{24}$$

where the matrix JA is the chosen normal form and $\xi = (D(c))^{-1}\gamma$.

Lemma 1 can be verified by direct calculation.

Note that equation (1) and system (24) are equivalent. However, the nonlinear system (24) obtained via the change (23) is not necessarily Hamiltonian.

Recall that the vector c is chosen only from the observability condition of the linear system (13). This provides much freedom when choosing the vector c. As it turns out, under some additional conditions imposed on the vector c, the nonlinear system (24) will be Hamiltonian. In particular, we have the following result.

Lemma 2. *Assume that the vector c is chosen based on two requirements:*

- *The linear system* (13) *is observable.*
- *For some real number* α,

$$\gamma = \alpha D(c)Jc, \tag{25}$$

where $D(c)$ *is the observability matrix of system* (13), γ *is the vector* (21), *and* J *is the matrix* (5).

Then the change of variables (23) *reduces the nonlinear equation* (1) *to a Hamiltonian system of the form* (24) *with the Hamiltonian*

$$H(x) = \frac{1}{2}(Ax, x) + \alpha F((x, c)), \tag{26}$$

where $F(y)$ *is the primitive of the function* $f(y)$, *i.e.*, $F'(y) = f(y)$.

Remark 2. Equality (25) in expanded form comes to a system of n linear algebraic equations with the $2n$ unknowns

$$\alpha c_1^2,\ \alpha c_2^2,\ \ldots,\ \alpha c_{2n}^2$$

and the parameter α. These equations include the coefficients determining the kind of the chosen normal form. As a result, the system of equations (25) is solvable only for one choice of the normal form. In other words, in contrast to the linear problem, the nonlinear one has a uniquely determined kind of the normal form of the Hamiltonian system constructed. This fact will be proved below for systems with two degrees of freedom.

Thus, we have the following result.

Theorem 3. *Assume that a possible normal form* JA *is chosen in accordance with the properties of the roots of the equation* $L(p) = 0$. *Let the vector* c *be chosen so that:*

1) *The linear system* (13) *is observable.*
2) *Equality* (25) *holds for some* α.

Then the change of variables (23) *reduces equation* (1) *to the equivalent Hamiltonian system* (24) *with the Hamiltonian* (26), *and the kind of its normal form is uniquely determined.*

4.2. Equations with Two Degrees of Freedom

As a basic application, consider the fourth-order equation

$$L\left(\frac{d}{dt}\right) y = M\left(\frac{d}{dt}\right) f(y), \tag{27}$$

where

$$L(p) = p^4 + ap^2 + b, \quad M(p) = b_0 p^2 + b_2 \tag{28}$$

are coprime real polynomials and $f(y)$ is a scalar continuous function. Equations of the form (27) are often called *equations with two degrees of freedom.*

As in Section 3.3, by assumption, the coefficients a and b of the polynomial $L(p)$ satisfy (15). Hence, all the four roots of the polynomial $L(p)$ are pure imaginary of the form $\pm i\omega_1,\ \pm i\omega_2$, where the numbers $\omega_1 > 0$ and $\omega_2 > 0$ are given by (16). We will construct an equivalent Hamiltonian system for equation (27) using Theorem 3.

As noted in Section 3.3, two different normal forms of the desired Hamiltonian system correspond to the polynomial $L(p)$, namely, the matrices (8) for $\sigma = 1$ and $\sigma = -1$. By analogy with Section 3.3, we choose a vector $c = (c_1, c_2, 0, 0, 0)$ such that $c_1 c_2 \neq 0$. In this case, the linear system (13) is observable.

It remains to ensure condition 2) of Theorem 3, i.e., choose the vector c so that equality (25) holds. In this equality, $D(c)$ is the matrix (18), and the four-dimensional vector γ is given by (21) and (22) with respect to equation (27):

$$\gamma = \begin{bmatrix} 0 \\ \gamma_2 \\ 0 \\ \gamma_4 \end{bmatrix} = \begin{bmatrix} 0 \\ b_0 \\ 0 \\ b_2 - ab_0 \end{bmatrix}.$$

Therefore, equality (25) comes to the system of two equations

$$\begin{cases} \alpha(\omega_1 c_1^2 + \sigma\omega_2 c_2^2) = -\gamma_2 \\ \alpha(\omega_1^3 c_1^2 + \sigma\omega_2^3 c_2^2) = \gamma_4 \end{cases}$$

with the unknowns αc_1^2 and αc_2^2. Hence, we obtain

$$\alpha c_1^2 = \frac{\omega_2^2\gamma_2 + \gamma_4}{\omega_1(\omega_1^2 - \omega_2^2)}, \quad \alpha c_2^2 = -\frac{\omega_1^2\gamma_2 + \gamma_4}{\sigma\omega_2(\omega_1^2 - \omega_2^2)}.$$

Due to the coprimeness of the polynomials (28),

$$(\omega_1^2\gamma_2 + \gamma_4)(\omega_2^2\gamma_2 + \gamma_4) \neq 0.$$

Therefore, $\alpha \neq 0$ and

$$\left(\frac{c_1}{c_2}\right)^2 = -\sigma\frac{\omega_2}{\omega_1}\kappa,$$

where

$$\kappa = \frac{\omega_2^2\gamma_2 + \gamma_4}{\omega_1^2\gamma_2 + \gamma_4}. \tag{29}$$

Thus, equation (25) is solvable only for $\sigma = 1$ (if $\kappa < 0$) or only for $\sigma = -1$ (if $\kappa > 0$).

Let $\kappa < 0$ ($\kappa > 0$). In this case, the following values can be taken as the solution of equation (25):

$$c_1 = 1, \quad c_2 = \sqrt{-\frac{\omega_1}{\kappa\omega_2}} \quad \left(c_2 = \sqrt{\frac{\omega_1}{\kappa\omega_2}}\right), \quad \alpha = \frac{\omega_2^2\gamma_2 + \gamma_4}{\omega_1(\omega_1^2 - \omega_2^2)}. \tag{30}$$

In other words, the following result has been established.

Theorem 4. *Assume that $\kappa < 0$ ($\kappa > 0$). Let the numbers α, c_1, and c_2 be given by* (30). *Then the change of variables* (23) *reduces equation* (27) *to the equivalent Hamiltonian system* (24) *with the Hamiltonian* (26):

$$H(x) = \frac{1}{2}(Ax, x) + \alpha F(x_1 c_1 + x_2 c_2),$$

where $F(y)$ is the primitive of the function $f(y)$, i.e., $F'(y) = f(y)$. In addition, the kind of the normal form (8) *is uniquely determined: $\sigma = 1$ in the case $\kappa < 0$ ($\sigma = -1$ in the case $\kappa > 0$).*

EXAMPLE 2

Consider equation (27) of the form

$$y'''' + 5y'' + 4y = (f(y))'' + 3f(y). \tag{31}$$

In other words, we have the polynomials (28) with $a = 5$, $b = 4$, $b_0 = 1$, and $b_2 = 3$. Then $\omega_1 = 2$ and $\omega_2 = 1$, and the vector γ is $\gamma = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix}$, i.e., $\gamma_2 = 1$ and $\gamma_4 = -2$.

Formula (29) yields $\kappa = -1/2 < 0$. By Theorem 4, the kind of the normal form (8) is uniquely determined: $\sigma = 1$. Next, the values (30) are $c_1 = 1$, $c_2 = 2$, and $\alpha = -1/6$.

According to Theorem 4, the change of variables (23) with the matrix $D(c)$ (18), $\omega_1 = 2$, $\omega_2 = 1$, $\sigma = 1$, $c_1 = 1$, and $c_2 = 2$ reduces equation (31) to the equivalent Hamiltonian system (24) with the matrix JA (8) with $\omega_1 = 2$, $\omega_2 = 1$, $\sigma = 1$, and the vector $\xi = (D(c))^{-1}\gamma$ equal to $\xi = \begin{bmatrix} 0 \\ 0 \\ 1/6 \\ 1/3 \end{bmatrix}$.

The Hamiltonian of this system is

$$H(x) = \frac{2x_1^2 + x_2^2 + 2x_3^2 + x_4^2}{2} - \frac{1}{6}F(x_1 + 2x_2).$$

5. CONCLUSIONS

This paper has proposed new approaches to constructing equivalent Hamiltonian systems for linear and nonlinear Lurie equations (differential equations containing derivatives of even orders only). The approaches are based on the transition from the linear part of the Lurie equation to the normal forms of the corresponding Hamiltonian systems, with a subsequent transformation of the resulting system. This scheme does not require complex and cumbersome transformations of the original equation. It has been demonstrated that, in the linear case, the problem of constructing equivalent Hamiltonian systems can lead to qualitatively different systems. At the same time, for nonlinear systems, the above problem is uniquely solvable in a natural sense. The Appendix contains similar results in a general formulation (without requiring that the original equations contain derivatives of even orders only). The main results have been reduced to computational formulas and algorithms.

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APPENDIX

Auxiliary Constructs

The proofs of the main theoretical results of this paper are based on the following auxiliary assertions of a general nature. They concern not only Hamiltonian systems and are of independent interest.

Consider a system described by the nth-order differential equation

$$L\left(\frac{d}{dt}\right) y = M\left(\frac{d}{dt}\right) u(t), \tag{A.1}$$

where

$$\begin{gathered} L(p) = p^n + a_1 p^{n-1} + \ldots + a_{n-1}p + a_n, \\ M(p) = b_0 p^m + b_1 p^{m-1} + \ldots + b_m \end{gathered} \tag{A.2}$$

are coprime real polynomials of degrees n and m ($n > m \geqslant 0$).

For equation (A.1), it is required to construct an equivalent system described by the equations

$$x' = Ax + \xi u(t), \quad y = (x(t), c), \tag{A.3}$$

where A is a square matrix of order n, $\xi, c \in R^n$ are fixed vectors, and the symbol (x, c) indicates the inner product of vectors x and c from R^n. The inverse problem is to construct from system (A.3) an equivalent system described by the differential equation (A.1).

The simplest transition is from (A.1) to the equivalent system

$$z' = A_0 z + \gamma u(t), \quad y = (z(t), c_0), \tag{A.4}$$

where $c_0 = (1, 0, 0, \ldots, 0)$,

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_2 & -a_1 \end{bmatrix}, \quad z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}, \quad \gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix},$$

and the coordinates of the vector γ are given by

$$\begin{gathered} \gamma_1 = \gamma_2 = \ldots = \gamma_{n-m-1} = 0, \quad \gamma_{n-m} = b_0, \quad \gamma_{n-m+1} + \gamma_{n-m} a_1 = b_1, \\ \ldots, \; \gamma_n + \gamma_{n-1} a_1 + \ldots + \gamma_{n-m} a_m = b_m. \end{gathered} \tag{A.5}$$

Direct calculation shows that the transition from equation (A.1) to system (A.4) can be implemented via the change of variables $z = \tilde{y} - T\tilde{u}$, where

$$\tilde{y} = \begin{bmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{bmatrix}, \quad \tilde{u} = \begin{bmatrix} u \\ u' \\ \vdots \\ u^{(n-2)} \end{bmatrix}, \tag{A.6}$$

and the rectangular matrix T of dimensions $n \times (n-1)$ has the form

$$T = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ \gamma_1 & 0 & 0 & \ldots & 0 \\ \gamma_2 & \gamma_1 & 0 & \ldots & 0 \\ & & & \ddots & \\ \gamma_{n-1} & \gamma_{n-2} & \gamma_{n-3} & \cdots & \gamma_1 \end{bmatrix}. \tag{A.7}$$

Similar problems arise for nonlinear systems. In them, the analog of equation (A.1) is a nonlinear feedback system described by

$$L\left(\frac{d}{dt}\right)y = M\left(\frac{d}{dt}\right)f(y),$$

where $L(p)$ and $M(p)$ are the polynomials (A.2) and $f(y)$ is a scalar continuous function. The analog of system (A.3) is the one described by

$$x' = Ax + \xi f(y), \quad y = (x(t), c).$$

Various issues related to these problems were discussed in many works. Let us emphasize the fundamental monograph [8] with a detailed analysis of basic concepts ("system," "equivalence," "transfer function", etc.) and, moreover, constructive methods for designing equivalent systems (within the linear theory).

To study the problems formulated here, we consider the following systems described by input-output-state equations:

- system $\mathcal{A}$ (A.1),
- system $\mathcal{B}$ described by the equations

$$x' = Ax + \xi u(t), \quad w = (x(t), c), \tag{A.8}$$

- system $\mathcal{C}$ described by the equations

$$z' = A_0 z + \gamma u(t), \quad v = (z(t), c_0). \tag{A.9}$$

Note that systems (A.8) and (A.9) are the same systems (A.3) and (A.4). They are presented in a new form only to avoid confusion with the notation of the outputs of the systems under consideration.

Let the input space of systems $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be the set C^m-smooth functions $u(t)$, and let their state space be the space R^n. For a given input $u(t)$ and a given initial state $\tilde{y}_0 = (y_0, y_1, \ldots, y_{n-1})$ (at the time $t = 0$), we define the output $y(t)$ of system $\mathcal{A}$ as the solution of the Cauchy problem

$$\begin{cases} L\left(\frac{d}{dt}\right)y = M\left(\frac{d}{dt}\right)u(t) \\ y(0) = y_0,\ y'(0) = y_1,\ \ldots,\ y^{(n-1)}(0) = y_{n-1}. \end{cases}$$

For a given input $u(t)$ and a given initial state $x_0 \in R^n$ (at the time $t = 0$), we define the output $w(t)$ of system $\mathcal{B}$ by the equality $w(t) = (x(t), c)$, where $x(t)$ is the solution of the Cauchy problem

$$\begin{cases} x' = Ax + \xi u(t) \\ x(0) = x_0. \end{cases}$$

The output $v(t)$ of system $\mathcal{C}$ is defined by analogy.

The following assertions are true.

Theorem 5. *Systems $\mathcal{A}$ and $\mathcal{C}$ are equivalent.*

Theorem 6. *Systems $\mathcal{B}$ and $\mathcal{C}$ are equivalent iff system $\mathcal{B}$ is observable, $A_0 = DAD^{-1}$, and $\gamma = D\xi$, where D denotes the observability matrix of system $\mathcal{B}$ and the vector γ consists of the coordinates* (A.5).

Assume that systems $\mathcal{B}$ and $\mathcal{C}$ are equivalent. Then system (A.4) *is reducible to system* (A.3) *via the nondegenerate change of variables $x = D^{-1}z$.*

Theorem 7. *Systems $\mathcal{A}$ and $\mathcal{B}$ are equivalent iff system $\mathcal{B}$ is observable, $A_0 = DAD^{-1}$, and $\gamma = D\xi$.*

Assume that systems $\mathcal{A}$ and $\mathcal{B}$ are equivalent. Then equation (A.1) *is reducible to system* (A.3) *via the change of variables*

$$x = D^{-1}(\tilde{y} - T\tilde{u}),$$

where the matrix T and the vectors $\tilde{y}$ and $\tilde{u}$ are given by (A.7) *and* (A.6)*, respectively.*

Theorem 5 is a well-known result; for example, see [2, 7, 8]. The validity of Theorem 7 follows from Theorems 5 and 6. Theorem 6 is established by standard methods of systems theory.

Proof of Theorem 1. *Necessity.* Let equation (9) and the Hamiltonian system (13) be equivalent. Then, by Theorem 7, system (13) is observable and the equality $A_0 = D(JA)D^{-1}$ holds with the matrix A_0 (12) and the observability matrix D of system (13).

Sufficiency. Let system (13) be observable. It is required to prove the equivalence of equation (9) and the Hamiltonian system (13). For this purpose, we show that the output $y(t) = (x(t), c)$ of system (13) is also that of equation (9) under an initial state y_0 such that $y_0 = (x(0), c)$ and, conversely, that each output $y(t)$ of equation (9) is also the output of system (13) under an initial state x_0 such that $y_0 = (x_0, c)$.

Let us restrict the considerations to the case $n = 2$ (i.e., system (13) is four-dimensional). Then equation (9) takes the form (14) and, therefore, $L(p) = p^4 + ap^2 + b$.

For the output $y(t) = (x(t), c)$ of system (13), we have

$$y' = (x', c) = (x, A^*c), \quad y'' = (x, (A^*)^2c), \quad y'''' = (x, (A^*)^4c).$$

Hence,

$$y'''' + ay'' + by = (x, (A^*)^4c) + (x, (A^*)^2c)a + (x, c)b = (x, [(A^*)^4 + a(A^*)^2 + bI]c) = 0,$$

since the matrix A (and, consequently, the transposed matrix A^*) satisfies its characteristic equation $p^4 + ap^2 + b = 0$. Thus, the function $y(t) = (x(t), c)$ is the solution of equation (9).

Now, let $y(t)$ be the output of equation (14); the corresponding initial state is $\tilde{y}_0 = (y_0, y_1, y_2, y_3)$. We determine the initial state x_0 of the four-dimensional system (13) from the system of equations

$$(x_0, c) = y_0, \quad (x_0, A^*c) = y_1, \quad (x_0, (A^*)^2c) = y_2, \quad (x_0, (A^*)^3c) = y_3$$

or (which is the same) from the equation $D(c)x_0 = \tilde{y}_0$. Due to the observability of system (13), this equation has the unique solution $x_0 = (D(c))^{-1}\tilde{y}_0$. Obviously, the output of system (13) under this initial state x_0 coincides with the function $y(t)$.

The proof of Theorem 1 is complete.

Proof of Theorem 2. This result is immediate from Theorem 7.

Proof of Lemma 2. According to Lemma 1, the change of variables (23) reduces equation (1) to system (24). To establish Lemma 2, it remains to show that the function (26) is the Hamiltonian of system (24), i.e., the validity of the relation

$$J\nabla H(x) = JAx + \xi f((c, x)).$$

Since $J\nabla H(x) = JAx + \alpha J\nabla F((x, c))$, we have to verify the equality

$$\alpha J\nabla F((x, c)) = \xi f((c, x)).$$

We have $\nabla F((x, c)) = f((c, x))c$, which yields $J\nabla F((x, c)) = f((c, x))Jc$. Thus, it is necessary to show $\alpha Jc = \xi$. In turn, this equality follows from (25) and the relation $\xi = D^{-1}\gamma$ (see Lemma 1).

The proof of Lemma 2 is complete.

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